THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 3

May 29, 2017

Name:

1. (50 points) Find the "standard forms" of the following quadratic equations and describe their solution sets.

$$4x^2 - 4xy + y^2 + y = -1$$

Solution

The matrix associated to the quadratic form is

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} .$$

Its determinant vanishes, so the curve is a parabola. To remove the mixed term, we first note that its eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = 5$.

For each of the above eigenvalues, we solve for an associated eigenvector:

For λ_1, the unit eigenvector is given by $(1/\sqrt{5}, -2/\sqrt{5})$. For λ_2, the unit eigenvector is given by $(-2/\sqrt{5}, 1/\sqrt{5})$. Therefore, by the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} ,$$

we have

$$\begin{aligned} 0 = 4x^2 - 4xy + y^2 + y + 1 &= 5y'^2 + \frac{1}{\sqrt{5}}(2x' + y') + 1 \\ &= (\sqrt{5}y' + \frac{1}{2\sqrt{25}})^2 + \frac{2}{\sqrt{5}}x' + \frac{99}{100} \end{aligned}$$

Finally, let $u = \frac{2}{\sqrt{5}}x'$ and $v = \sqrt{5}y' + \frac{1}{2\sqrt{25}}$, we have

$$0 = u^2 + v + c.$$

where $c = 99/100$.

2. (50 points) Show that the polar equation for the circle centered at $(a, 0)$ with radius a, where $a > 0$, is given by

$$\rho(\theta) = 2a\cos\theta, \quad \theta \in (-\pi/2, \pi/2] .$$

Solution Equation of such a circle in rectangular coordinates is given by

$$(x - a)^2 + y^2 = a^2$$

which simplifies to

$$x^2 + y^2 = 2ax$$

Apply the polar coordinate change of variables $x = \rho\cos\theta, y = \rho\sin\theta$, we have

$$(\rho\cos\theta)^2 + (\rho\sin\theta)^2 = 2a\rho\cos\theta$$

which simplifies to

$$\rho^2 = 2a\rho\cos\theta$$

Since $\rho > 0$, it further simplifies to

$$\rho = 2a\cos\theta$$

which is the polar equation of the circle.